matrix repre:



$a$: $\quad a|n\rangle = \sqrt{n}\,|n-1\rangle$

$$\langle m|a|n\rangle = \langle m|\sqrt{n}\,|n-1\rangle = \sqrt{n}\,\langle m|n-1\rangle = \sqrt{n}\,\delta_{m,n-1}$$

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & \sqrt{2} & 0 \\ 2 & 0 & 0 & 0 & \sqrt{3} \\ 3 & 0 & 0 & 0 & 0 \quad \sqrt{4} \\ & & & & \ddots \end{array}$$

$$\langle m|a^\dagger|n\rangle = \langle m|\sqrt{n+1}\,|n+1\rangle = \sqrt{n+1}\,\langle m|n+1\rangle = \sqrt{n+1}\,\delta_{m,n+1}$$

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{2} & 0 & 0 \\ 3 & 0 & 0 & \sqrt{3} & 0 \\ & & & & \sqrt{4} \ \ddots \end{array}$$

$$a a^\dagger: \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{3} & \cdot \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & \cdot \\ 0 & 0 & \sqrt{3} & 0 \\ & & \vdots \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & 2 & \\ 0 & & 3 \\ & & & 4 \ \ddots \end{pmatrix}$$

$$a^\dagger a = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 3 \\ & & \vdots \end{pmatrix}$$

$$[a,a^\dagger] = a a^\dagger - a^\dagger a = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \ \ddots \end{pmatrix} - \begin{pmatrix} 0 & & \\ & 1 & \\ & & 2 \ \ddots \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \ \ddots \end{pmatrix} = \mathbb{1}$$

$$X = \frac{\alpha}{\sqrt{2}}(a + a^\dagger)$$

$$\langle m|X|n\rangle = \frac{\alpha}{\sqrt{2}}\Big[\overbrace{\langle m|a|n\rangle}^{a} + \overbrace{\langle m|a^\dagger|n\rangle}^{a^\dagger}\Big]$$

$$= \frac{\alpha}{\sqrt{2}}\Big[\sqrt{n}\,\delta_{m,n-1} + \sqrt{n+1}\,\delta_{m,n+1}\Big]$$

$$X = \frac{\alpha}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \cdots \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} \\ & & & \sqrt{4} & \ddots \end{pmatrix}$$

(Hermitian btw)

$$\begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ X^2 = \underbrace{\frac{\alpha^2}{2}}_{\alpha^2 = \frac{\hbar}{m\omega}} & \begin{pmatrix} 1 & 0 & \sqrt{2} & 0 \\ 0 & 3 & 0 & \sqrt{6} \\ \sqrt{2} & 0 & 5 & 0 \\ 0 & \sqrt{6} & 0 & 7 \\ & & & & \ddots \end{pmatrix} \begin{array}{l} \\ \leftarrow \sqrt{n}\sqrt{n+2} \\ \\ \nwarrow 2n+1 \end{array} \end{array}$$

$$p = \frac{i}{\sqrt{2}}\sqrt{\hbar m\omega}\,(a^\dagger - a) = \frac{i}{\sqrt{2}}\sqrt{\hbar m\omega} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -\sqrt{2} & 0 & \cdots \\ 0 & \sqrt{2} & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 & \ddots \end{pmatrix}$$

$$p^2 = -\frac{\hbar m\omega}{2} \begin{pmatrix} -1 & 0 & \sqrt{2} & 0 \\ 0 & -3 & 0 & \sqrt{6} \\ \sqrt{2} & 0 & -5 & 0 \\ 0 & \sqrt{6} & 0 & \ddots \end{pmatrix} = \frac{\hbar m\omega}{2} \begin{pmatrix} 1 & 0 & -\sqrt{2} & \\ 0 & 3 & 0 & -\sqrt{6} \\ -\sqrt{2} & 0 & 5 & 0 \\ & -\sqrt{6} & 0 & 7 \ \ddots \end{pmatrix}$$

$$\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \frac{\hbar\omega}{4} \begin{pmatrix} 1 & 0 & -\sqrt{2} & \\ 0 & 3 & 0 & -\sqrt{6} \\ -\sqrt{2} & 0 & 5 & \ddots \\ & -\sqrt{6} & & \end{pmatrix} + \frac{1}{4}\hbar\omega \begin{pmatrix} 1 & 0 & \sqrt{2} & 0 \\ 0 & 3 & 0 & \sqrt{6} \\ \sqrt{2} & 0 & 5 & 0 \\ 0 & \sqrt{6} & 0 & 7 \\ & & \vdots \end{pmatrix} = \frac{\hbar\omega}{2} \begin{pmatrix} 1 & & 0 \\ & 3 & \\ 0 & & 5 & \\ & & & 7 \ \ddots \end{pmatrix}$$

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$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 - |e| x E$$

čtverec

$$\frac{1}{2} m\omega^2 x^2 - e x E + \frac{e^2 E^2}{2m\omega^2} - \frac{e^2 E^2}{2m\omega^2}$$

$$A^2 = \frac{1}{2} m\omega^2$$

$$2AB = -eE$$

$$2\frac{1}{\sqrt{2}}\sqrt{m}\,\omega B = -eE$$

$$B = -\frac{eE}{\sqrt{m}\,\omega\sqrt{2}}$$

$$\frac{1}{2} m\omega^2 \left(x^2 - \frac{2ex}{m\omega^2} + \frac{e^2 E^2}{m^2\omega^4}\right) = \frac{1}{2} m\omega^2 \left(x - \frac{eE}{m\omega^2}\right)^2$$

$$\to E = \frac{1}{2}\hbar\omega - \frac{e^2 E^2}{2m\omega^2}$$

$$\to H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 \left(x - \frac{eE}{m\omega^2}\right)$$ ok

$$H_0 = \frac{1}{2}\hbar\omega$$

$$\frac{3}{2}\hbar\omega$$

$$\frac{5}{2}\hbar\omega$$

$$V' = -exE$$

$$V' = -\frac{eE\alpha}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \\ & & & & \ddots \end{pmatrix}$$

$$H = \begin{pmatrix} \frac{1}{2}\hbar\omega & -\frac{eE\alpha}{\sqrt{2}} & 0 \\ -\frac{eE\alpha}{\sqrt{2}} & \frac{3}{2}\hbar\omega & -eE\alpha \\ 0 & -eE\alpha & \frac{5}{2}\hbar\omega \end{pmatrix}$$

$$A = \hbar\omega \qquad B = \frac{eE\alpha}{\sqrt{2}} \ldots$$

$$\begin{pmatrix} \frac{1}{2}A - \lambda & -B \\ -B & \frac{3}{2}A - \lambda \end{pmatrix} \to \mathrm{Det} = \cancel{\frac{3}{4}A^2 - B^2 = \left(\frac{\sqrt{3}}{2}A - B\right)\left(\frac{\sqrt{3}}{2}A + B\right)}$$

$$\left(\frac{1}{2}A - \lambda\right)\left(\frac{3}{2}A - \lambda\right) - B^2 = \frac{3}{4}A^2 - 2A\lambda + \lambda^2 - B^2$$

$$\lambda^2 - 2A\lambda + \frac{3}{4}A^2 - B^2$$

$$D = 4A^2 - 4\left(\frac{3}{4}A^2 - B^2\right) = 4A^2 - 3A^2 + 4B^2$$

$$= A^2 + 4B^2$$

$$\lambda_{1,2} = \frac{+2A \pm \sqrt{A^2 + 4B^2}}{2}$$

$$= A \pm \frac{1}{2}\sqrt{A^2 + 4B^2} = A \pm \frac{1}{2} A\sqrt{1 + \frac{4B^2}{A^2}}$$

$$\approx A \pm \frac{1}{2} A\left(1 + \frac{4B^2}{2A^2} - \frac{16B^4}{8A^4}\right) \ldots$$

$$\alpha = \sqrt{\frac{\hbar}{m\omega}}$$

$$\lambda_- = A - \frac{1}{2}A - \frac{4B^2}{4A} + \frac{B^4}{A^3} = \hbar\omega - \frac{1}{2}\hbar\omega - \frac{\frac{e^2E^2\alpha^2}{2}}{\hbar\omega} = \frac{1}{2}\hbar\omega - \frac{e^2E^2}{2m\omega^2}$$

[illegible]...

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T.P.2

$$H = \begin{pmatrix} \frac{1}{2}\hbar\omega & -\frac{eEd}{\sqrt{2}} \\ -\frac{eEd}{\sqrt{2}} & \frac{3}{2}\hbar\omega \end{pmatrix} = \begin{pmatrix} \frac{1}{2}A & -B \\ -B & \frac{3}{2}A \end{pmatrix}$$

$$\begin{vmatrix} \frac{1}{2}A-\lambda & -B \\ -B & \frac{3}{2}A-\lambda \end{vmatrix} = \left(\tfrac{1}{2}A-\lambda\right)\left(\tfrac{3}{2}A-\lambda\right)+B^2$$

$$= \frac{3}{4}A^2+\lambda^2-2A\lambda+B^2 = \lambda^2-2A\lambda+\frac{3}{4}A^2+B^2 \quad \text{OK}$$

$$D = 4A^2-4\left(\tfrac{3}{4}A^2+B^2\right) = 4A^2-3A^2+4B^2 = A^2+4B^2$$

$$\lambda_{1,2} = \frac{2A\pm\sqrt{A^2+4B^2}}{2} = A\pm\frac{1}{2}\sqrt{A^2+4B^2} = A\pm\frac{1}{2}A\sqrt{1+\frac{4B^2}{A^2}}$$

$$= A\pm\frac{1}{2}A\left(1+\frac{4B^2}{2A^2}-\frac{16B^4}{8A^4}\dots\right)$$

$$= A\pm\left(\frac{1}{2}A+\frac{B^2}{A}-\frac{B^4}{A^3}\dots\right)$$

$$\lambda_- = A-\left(\frac{1}{2}A+\frac{B^2}{A}-\frac{B^4}{A^3}\right) = \frac{1}{2}A-\frac{B^2}{A}+\frac{B^4}{A^3}+\dots$$

$A=\hbar\omega \quad B=\frac{eEd}{\sqrt{2}}$

$$= \frac{1}{2}\hbar\omega-\frac{e^2E^2d^2}{2\hbar\omega}+\frac{e^4E^4d^4}{2\hbar^3\omega^3}$$

$d^2=\frac{\hbar}{m\omega}$

$$= \frac{1}{2}\hbar\omega \underbrace{-\frac{e^2E^2}{2m\omega^2}}_{\text{OK}}+\underbrace{\frac{e^4E^4}{2\hbar m^2\omega^5}+\,?}_{\text{actually errors}}$$

cancelled by higher-n contributions

$$\Delta E^{(2)} = \sum_i \frac{|\langle 0|V'|i\rangle|^2}{\varepsilon_0-\varepsilon_i} = \sum \frac{|\langle 0|V'|1\rangle|^2}{\frac{1}{2}\hbar\omega-\frac{3}{2}\hbar\omega} =$$

$$= \frac{\left|-\frac{eEd}{\sqrt{2}}\right|^2}{-\hbar\omega} = -\frac{e^2E^2d^2}{2\hbar\omega} = -\frac{e^2E^2}{2m\omega^2} \quad \text{OK}$$

$-eEd$

$-\frac{eEd}{\sqrt{2}}$

$$\Delta E^{(4)} = \frac{\langle 0|V'|1\rangle\langle 1|V'|2\rangle\langle 2|V'|1\rangle\langle 1|V'|0\rangle}{(\varepsilon_0-\varepsilon_1)^2(\varepsilon_0-\varepsilon_2)} + \dots = \frac{e^4E^4d^4}{4\hbar^3\omega^3} = \frac{e^4E^4\hbar^2}{4\hbar^3\omega^3\, m^2\omega^2}$$

například

$$= \frac{e^4E^4}{4\omega^5\hbar m^2}$$

...

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$$f(x,x_0) = \frac{1}{\sqrt{\alpha\sqrt{\pi}}}\, e^{-\frac{(x-x_0)^2}{2\alpha^2}}$$

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 - exE \qquad \frac{p^2}{2m} = \frac{(-i\hbar D)^2}{2m} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$$

$$\langle T\rangle(x_0) = \frac{1}{\alpha\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-x_0)^2}{2\alpha^2}} \left(-\frac{\hbar^2}{2m}\right) \frac{d^2}{dx^2}\, e^{-\frac{(x-x_0)^2}{2\alpha^2}}$$

$$= \frac{1}{\alpha\sqrt{\pi}}\left(-\frac{\hbar^2}{2m}\right) \int_{-\infty}^{\infty} e^{-\frac{(x-x_0)^2}{2\alpha^2}} \frac{d}{dx}\left[-\frac{(x-x_0)}{\alpha^2} e^{-\frac{(x-x_0)^2}{2\alpha^2}}\right]$$

$$= \qquad \left[-\frac{1}{\alpha^2} + \frac{(x-x_0)^2}{\alpha^4}\right] e^{-\frac{(x-x_0)^2}{2\alpha^2}}$$

$$= \frac{1}{\alpha\sqrt{\pi}}\left(-\frac{\hbar^2}{2m}\right) \int_{-\infty}^{\infty} e^{-\frac{(x-x_0)^2}{\alpha^2}} \left[-\frac{1}{\alpha^2} + \frac{(x-x_0)^2}{\alpha^4}\right]$$

$$y = x - x_0 \qquad dy = dx \qquad e^{-y^2} \to \alpha\sqrt{\pi}$$

$$= \frac{1}{\alpha\sqrt{\pi}}\left(-\frac{\hbar^2}{2m}\right)\left[-\frac{\alpha\sqrt{\pi}}{\alpha^2} + \frac{\alpha^2}{2\alpha^4}\alpha\sqrt{\pi}\right] \qquad x^2 e^{-()} \to \frac{\alpha^2}{2}\alpha\sqrt{\pi}$$

$$= -\frac{\hbar^2}{2m}\left[-\frac{1}{\alpha^2} + \frac{1}{2\alpha^2}\right] = \frac{1}{2}\frac{m\omega}{\hbar}\frac{\hbar^2}{2m} = \frac{1}{4}\hbar\omega \qquad \text{OK}$$

← nezávisí na $x_0$, dáno tvarem $\psi$

$$\langle x^2\rangle = \frac{1}{\alpha\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-x_0)^2}{\alpha^2}} x^2\, dx = \frac{1}{\alpha\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{\alpha^2}} (y+x_0)^2\, dy$$

$$= \frac{1}{\alpha\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{\alpha^2}} \left(y^2 + 2x_0 y + x_0^2\right) dy$$

$$= \frac{1}{\alpha\sqrt{\pi}}\left[\frac{\alpha^2}{2}\alpha\sqrt{\pi} + x_0^2\,\alpha\sqrt{\pi}\right] = \frac{\alpha^2}{2} + x_0^2$$

$$\langle x\rangle = \frac{1}{\alpha\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-x_0)^2}{\alpha^2}} x\, dx = \frac{1}{\alpha\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{\alpha^2}} (y+x_0)\, dy$$

$$= \frac{1}{\alpha\sqrt{\pi}}\,\alpha\sqrt{\pi}\, x_0$$

$$\langle E\rangle_{x_0} = \frac{1}{4}\hbar\omega + \frac{1}{2} m\omega^2\Big(\overbrace{\frac{\alpha^2}{2}}^{\frac{\hbar}{m\omega}} + x_0^2\Big) - eEx_0 = \frac{1}{2}\hbar\omega + \frac{1}{2} m\omega^2 x_0^2 - eEx_0$$

$$\frac{d\langle E\rangle_{x_0}}{dx_0} = m\omega^2 x_0 - eE \overset{\text{extrema}}{=} 0$$

$$x_0 = \frac{eE}{m\omega^2} \qquad \text{OK}$$

$$E = \frac{1}{2}\hbar\omega + \frac{1}{2} m\omega^2 \frac{e^2E^2}{m^2\omega^4} - eE\frac{eE}{m\omega^2}$$

$$= \frac{1}{2}\hbar\omega + \frac{1}{2}\frac{e^2E^2}{m\omega^2} - \frac{e^2E^2}{m\omega^2}$$

$$= \frac{1}{2}\hbar\omega - \frac{1}{2}\frac{e^2E^2}{m\omega^2} \qquad \text{OK}$$

$$H_0 = \frac{p^2}{2m} + \frac{1}{2} m\omega x^2 - e\varepsilon E$$

$$f(x) = \frac{1}{\sqrt{1+c^2}} \frac{1}{\sqrt{\alpha\sqrt{\pi}}} \left(1 + \sqrt{2}\, c\, \frac{x}{\alpha}\right) e^{-\frac{x^2}{2\alpha^2}} = \frac{1}{\sqrt{1+c^2}} \left(|0\rangle + c|1\rangle\right)$$

$$E(c) = \langle f|\hat{H}|f\rangle = \frac{1}{1+c^2} \left(\langle 0| + c^*\langle 1|\right) H \left(|0\rangle + c|1\rangle\right)$$

assume real

$$= \frac{1}{1+c^2} \left(\left(\langle 0| + c\langle 1|\right)\left(H_0 + V'\right)\left(|0\rangle + c|1\rangle\right)\right)$$

$$= \frac{1}{1+c^2} \Big[\langle 0|H_0|0\rangle + c^2\langle 1|H_0|1\rangle + c\langle 0|H_0|1\rangle + c\langle 1|H_0|0\rangle$$
$$+ \langle 0|V'|0\rangle + c^2\langle 1|V'|1\rangle + c\langle 0|V'|1\rangle + c\langle 1|V'|0\rangle\Big]$$

$$= \frac{1}{1+c^2} \left[\frac{1}{2}\hbar\omega + c^2 \frac{3}{2}\hbar\omega + 0 + 0 + 0 + 0 + 2c \cdot \left(-\frac{eE\alpha}{\sqrt{2}}\right)\right]$$

$$\frac{1}{2}c^2\hbar\omega + c^2\hbar\omega$$

$$= \frac{1}{2}\hbar\omega + \underbrace{\frac{1}{1+c^2}\left[c^2\hbar\omega + 2c\left(-\frac{eE\alpha}{\sqrt{2}}\right)\right]}_{\Delta E}$$

$$\frac{d\Delta E}{dc} = -\frac{1}{(1+c)^2}\left[c^2\hbar\omega + 2c\left(-\frac{eE\alpha}{\sqrt{2}}\right)\right] + \frac{1}{1+c^2}\left(2c\hbar\omega + 2\left(-\frac{eE\alpha}{\sqrt{2}}\right)\right)$$

$$= -\frac{1}{(1+c^2)^2}\left[c^2\hbar\omega - 2c\frac{eE\alpha}{\sqrt{2}} + (1+c^2)\left(2c\hbar\omega - 2\frac{eE\alpha}{\sqrt{2}}\right)\right]$$

$$c^2\hbar\omega - 2c\frac{eE\alpha}{\sqrt{2}} + 2c\hbar\omega + 2c^3\hbar\omega - \frac{2cE\alpha}{\sqrt{2}} - 2\frac{c^2eE\alpha}{\sqrt{2}} \Big] = 0$$

... ?

mom. hybashi basics

$$\vec{L} = \vec{r}\times\vec{p} = -i\hbar\left(\underbrace{y\frac{d}{dz} - z\frac{d}{dy}}_{\hat L_x},\ \underbrace{z\frac{d}{dx} - x\frac{d}{dz}}_{\hat L_y},\ \underbrace{x\frac{d}{dy} - y\frac{d}{dx}}_{\hat L_z}\right) \quad (3D)$$

$$[\hat L_x,\hat L_y] = -\hbar^2\left[y\frac{d}{dz} - z\frac{d}{dy},\ z\frac{d}{dx} - x\frac{d}{dz}\right] = -\hbar^2\left[y\frac{d}{dz}, z\frac{d}{dx}\right] - \hbar^2\left[-z\frac{d}{dy}, x\frac{d}{dz}\right]$$

$$= -\hbar^2\left(y\frac{d}{dz}z\frac{d}{dx} - z\frac{d}{dx}y\frac{d}{dz}\right) + \hbar^2\left(z\frac{d}{dy}x\frac{d}{dz} - x\frac{d}{dz}z\frac{d}{dy}\right)$$

$$= -\hbar^2 y\frac{d}{dx}\left(\frac{d}{dz}z - z\frac{d}{dz}\right) + \hbar^2\frac{d}{dy}x\left(z\frac{d}{dz} - \frac{d}{dz}z\right)$$

$$\left(\frac{d}{dz}z - z\frac{d}{dz}\right)f(z) = \frac{d}{dz}zf(z) - z\frac{df}{dz} = f(z) + z\frac{df}{dz} - z\frac{df}{dz} = f(z) = 1\cdot f(z)$$

$$= -\hbar^2 y\frac{d}{dx} - \hbar^2\frac{d}{dy}x = i\hbar\, i\hbar\left(y\frac{d}{dx} - \frac{d}{dy}x\right) = i\hbar(yp_x - p_y x) = i\hbar L_z$$

sign?

$[x,p_x], \ldots \quad [x, L_y], \ldots \quad [L_z \ldots] \quad [L_x, L_y]$

$$\vec L = \vec r\times\vec p = (yp_z - zp_y,\ zp_x - xp_z,\ xp_y - yp_x)$$

$$[x,p_x]f = xp_xf - p_xxf = xp_xf - (-i\hbar)\frac{d}{dx}xf = x(-i\hbar)\frac{df}{dx} + i\hbar f + i\hbar x\frac{df}{dx} = i\hbar f$$

$$\Rightarrow [x,p_x] = i\hbar$$

$$[x,L_x] = [x, yp_z - zp_y] = 0$$

$$[x,L_y] = [x, zp_x - xp_z] = [x,zp_x] - \underbrace{[x,xp_z]}_{=0} = z[x,p_x] = i\hbar z$$

$$[x,L_z] = [x, xp_y - yp_x] = \underbrace{[x,xp_y]}_{=0} - [x,yp_x] = -y[x,p_x] = -i\hbar y$$

$$[p_x,L_y] = [p_x, zp_x - xp_z] = \underbrace{[p_x, zp_x]}_{=0} - [p_x, xp_z] = -p_z[p_x,x] = i\hbar p_z$$

$$[L_x,L_y] = [yp_z - zp_y,\ zp_x - xp_z] = [yp_z, zp_x] + [zp_y, xp_z] =$$

$$= yp_x\underbrace{[p_z,z]}_{-i\hbar} + p_y x\underbrace{[z,p_z]}_{i\hbar} = yp_x(-i\hbar) + i\hbar p_y x = i\hbar(\underbrace{xp_y - yp_x}_{L_z})$$

$\vdots$

$$= i\hbar L_z$$

$$L^2 = L_x^2+L_y^2+L_z^2$$

$$[L^2, L_x] = [L_x^2+L_y^2+L_z^2, L_x] = [L_y^2, L_x] + [L_z^2, L_x]$$

$$= L_y^2L_x - L_xL_y^2 + L_z^2L_x - L_xL_z^2$$

$$= L_y^2L_x - L_yL_xL_y + L_yL_xL_y - L_xL_yL_y + L_zL_zL_x - L_zL_xL_z + L_zL_xL_z - L_xL_z^2$$

$$= L_y\underbrace{[L_y,L_x]}_{-i\hbar L_z} + \underbrace{[L_y,L_x]}_{-i\hbar L_z}L_y + L_z\underbrace{[L_z,L_x]}_{i\hbar L_y} + \underbrace{[L_z,L_x]}_{i\hbar L_y}L_z$$

$$= -2i\hbar L_yL_z + 2i\hbar L_yL_z = 0$$