Plán: • $(x+\frac{d}{dx})^2$ nebo $\frac{d}{dx}x^2$ $[\frac{d}{dx}, x] = 1$ & expl. $f(x)$ UKM-2023-T2-1

• $\frac{1}{\sqrt{2\pi}}e^{in\phi}$ • ON báze

• Op $\left(\frac{d}{d\phi}e^{2i\phi}\right)$ upravit +, explicitně

matice $\frac{d}{dx}$, $e^{i\phi}$, působení napravo a nalevo + jako integrál explicitně

$\langle\ \rangle$ $e^{i\phi}$, $\frac{d}{d\phi}$, $\frac{d^2}{d\phi}$

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-in\phi}e^{in\phi}\,d\phi = \frac{1}{2\pi}\int_0^{2\pi} 1\,d\phi = 1 \quad \text{OK} \leftarrow \langle n|n\rangle = 1$$

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-in\phi}e^{im\phi}\,d\phi = \frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)\phi}\,d\phi = \delta_{m,n}$$

$\langle n|m\rangle = \delta_{nm}$

$$\left[\frac{d}{d\phi}, e^{i\phi}\right] = ie^{i\phi} \Rightarrow \frac{d}{d\phi}e^{i\phi} = ie^{i\phi} + e^{i\phi}\frac{d}{d\phi}$$

$$\frac{d}{d\phi}e^{2i\phi}f(\phi) = 2ie^{i2\phi}f(\phi) + e^{2i\phi}f'(\phi) = \cancel{2i\left(\tfrac{1}{2i}+\tfrac{d}{d\phi}\right)f(\phi)}$$

$$e^{2i\phi}\left(2i + \frac{d}{d\phi}\right)f(\phi)$$

$$\frac{d}{d\phi}e^{i\phi}e^{i\phi} = \left(ie^{i\phi} + e^{i\phi}\frac{d}{d\phi}\right)e^{i\phi} = ie^{2i\phi} + e^{i\phi}\left(ie^{i\phi} + e^{i\phi}\frac{d}{d\phi}\right)$$

$$= 2ie^{2i\phi} + e^{2i\phi}\frac{d}{d\phi} \quad \text{OK}$$

$$\begin{array}{ccccccccc} |0\rangle & |1\rangle & |-1\rangle & \dots \\ 0 & 1 & -1 & 2 & -2 & 3 & -3 & & \infty \end{array}$$

$\frac{1}{\sqrt{2\pi}}e^{in\phi}$ je ON báze → vektor $(\ldots)$

$$\sin(n\phi) = \frac{1}{2i}\left(e^{in\phi} - e^{-in\phi}\right)$$

normalizace

$f(\phi) = N\sin(n\phi)$

$$\langle f|f\rangle = 1 = N^2\int_0^{2\pi}\sin^2(n\phi)\,d\phi = N^2\cdot\frac{1}{2}\cdot 2\pi = N^2\pi$$

$$\Rightarrow N = \frac{1}{\sqrt{\pi}}$$

$$\underbrace{\frac{1}{\sqrt{\pi}}\sin(n\phi)}_{|S_n\rangle} = \frac{1}{2i\sqrt{\pi}}\left(e^{in\phi} - e^{-in\phi}\right) = \frac{1}{i\sqrt{2}}\Big(\underbrace{\frac{1}{\sqrt{2\pi}}e^{in\phi}}_{|n\rangle} - \underbrace{\frac{1}{\sqrt{2\pi}}e^{-in\phi}}_{|-n\rangle}\Big)$$

$\langle S_n|S_n\rangle = 1$?

$$\frac{1}{-i\sqrt{2}}\frac{1}{i\sqrt{2}}\left(\langle n| - \langle -n|\right)\left(|n\rangle - |-n\rangle\right) = -\frac{1}{i\cdot i}\frac{1}{2}\left[\langle n|n\rangle - \langle n|-n\rangle - \langle -n|n\rangle + \langle -n|-n\rangle\right]$$

$$= -\frac{1}{-1}\frac{1}{2}\left[1 - 0 - 0 + 1\right] = 1$$

$$\int_0^{2\pi}\frac{1}{-i\sqrt{2}}\frac{1}{i\sqrt{2}}\left[\frac{1}{\sqrt{2\pi}}e^{-in\phi} - \frac{1}{\sqrt{2\pi}}e^{in\phi}\right]\left[\frac{1}{\sqrt{2\pi}}e^{in\phi} - \frac{1}{\sqrt{2\pi}}e^{-in\phi}\right]d\phi =$$

$$= \frac{1}{2}\int_0^{2\pi}\frac{1}{2\pi}\left(e^0 - e^{-2in\phi} - e^{2in\phi} + e^0\right)d\phi = 1 \quad \text{OK}$$

$\hat{A} = \frac{d}{d\phi}$ → matice ? $\mathbb{A}$?

$$\mathbb{A}_{mn} = \langle m|\hat{A}|n\rangle$$
$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-im\phi}\frac{d}{d\phi} e^{in\phi}\,d\phi = \frac{1}{2\pi}\int_0^{2\pi} e^{-im\phi}\, in\, e^{in\phi}\,d\phi$$
$$= \frac{in}{2\pi}\int_0^{2\pi} e^{-im\phi} e^{in\phi}\,d\phi = in\,\delta_{mn} \rightarrow \mathbb{A} = \begin{pmatrix} 0 & & & & \\ & i & & 0 & \\ & & -i & & \\ & 0 & & 2i & \\ & & & & -2i \\ & & & & & \ddots \end{pmatrix}$$

$$A|n\rangle = \frac{d}{d\phi}\frac{1}{\sqrt{2\pi}}e^{in\phi} = \frac{1}{\sqrt{2\pi}}\, in\, e^{in\phi} = in|n\rangle$$ vl. fce/vektor

$$\hat{B} = e^{i\phi}$$
$$\mathbb{B}_{mn} = \langle m|\hat{B}|n\rangle$$
$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-im\phi} e^{i\phi} e^{in\phi}\,d\phi = \frac{1}{2\pi}\int_0^{2\pi} e^{i(n+1-m)\phi}\,d\phi = \delta_{n+1,m}$$
$$\underbrace{}_{n+1-m=0}$$

$$\begin{array}{c|c} & n \rightarrow \\ & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} m \\ 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & & 1 & \ddots \end{pmatrix} \end{array}$$

$$\begin{array}{c|c} & \begin{array}{ccccc} 0 & 1 & -1 & 2 & -2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ -1 \\ 2 \\ -2 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & \\ 1 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & \ddots \end{pmatrix} \end{array}$$ obrácené

$\cancel{\hat{A}}$

$$\hat{B}|n\rangle = e^{i\phi}\frac{1}{\sqrt{2\pi}}e^{in\phi} = \frac{1}{\sqrt{2\pi}}e^{i(n+1)\phi}$$ ← nejsou vl. fce/vektory

$$\hat{T} = -\frac{\hbar^2}{2I}\frac{d^2}{d\phi^2} = -\frac{\hbar^2}{2I}\hat{A}^2$$

~~matice~~ ~~jsou vl.~~ jsou $|n\rangle$ vl. stavy?

$$\mathbb{T} = -\frac{\hbar^2}{2I}\mathbb{A}^2 = -\frac{\hbar^2}{2I}\begin{pmatrix} 0 & & & & \\ & -1 & & 0 & \\ & & -1 & & \\ & 0 & & -4 & \\ & & & & -4 \end{pmatrix} = \frac{\hbar^2}{2I}\begin{pmatrix} 0 & & & & \\ & 1 & & 0 & \\ & 0 & 1 & & \\ & & & 4 & \\ & & & & 4 \end{pmatrix}$$

kv. rotor $|0\rangle$ základní stav

$\cancel{A}\ \hat{A}|n\rangle$

$\hat{A} = \frac{d}{d\phi}: \quad \frac{1}{\sqrt{2\pi}}\frac{d}{d\phi}e^{in\phi} = in\frac{1}{\sqrt{2\pi}}e^{in\phi}$

$\hat{A}|n\rangle = in|n\rangle \quad \leftarrow$ vlastní stav

$\hat{B} = e^{i\hat{\phi}}: \quad \frac{1}{\sqrt{2\pi}}e^{i\phi}e^{in\phi} = \frac{1}{\sqrt{2\pi}}e^{i(n+1)\phi} = |n+1\rangle \quad \leftarrow$ není vl. stav

$\hat{A}|s_n\rangle: \quad \frac{1}{\sqrt{\pi}}\frac{d}{d\phi}\sin(n\phi) = \frac{1}{\sqrt{\pi}}\cancel{\frac{d}{d}}\,n\cos(n\phi) = n|c_n\rangle$

$\hat{A}\hat{B}|n\rangle = \hat{A}|n+1\rangle = i(n+1)|n+1\rangle$

$\hat{B}\hat{A}|n\rangle = \hat{B}\,in|n\rangle = in|n+1\rangle$

$[\hat{A},\hat{B}]|n\rangle = i|n+1\rangle \qquad \hat{A}\hat{B}|n\rangle = i|n+1\rangle + \hat{B}\hat{A}|n\rangle$

$e^{-i\phi}$

matrix A: $\begin{pmatrix} -2i & & & & \\ & -i & & 0 & \\ & & 0 & & \\ & 0 & & i & \\ & & & & 2i \\ & & & & & \ddots \end{pmatrix}$

$$\begin{array}{c|c} & \begin{matrix} -2 & -1 & 0 & 1 & 2 \end{matrix} \\ \hline \begin{matrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{matrix} & \mathbb{B} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & 0 \\ & & 0 & 1 & \\ & 0 & & 0 & 1 \\ & & & & 0 \\ & & & & & \ddots \end{pmatrix} \end{array}$$

$\mathbb{A}\mathbb{B} \overset{5\times5}{=} \begin{pmatrix} 0 & -2i & & & \\ & 0 & -i & & 0 \\ & 0 & 0 & 0 & \\ & & & 0 & i \\ & & & & 0 \end{pmatrix} \times \qquad \mathbb{B}\mathbb{A} \overset{5\times5}{=} \begin{pmatrix} 0 & -i & & & \\ 0 & 0 & 0 & & 0 \\ & & 0 & i & \\ & & & 0 & 2i \\ & & & & 0 \end{pmatrix} \times$

$\mathbb{A}\mathbb{B} - \mathbb{B}\mathbb{A} = \begin{pmatrix} 0 & -i & & & \\ & 0 & -i & & \\ & & 0 & -i & \\ & 0 & & 0 & -i \\ & & & & 0 \end{pmatrix} = -i\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & 0 \\ & & 0 & 1 & \\ & 0 & & 0 & 1 \\ & & & & 0 \end{pmatrix} \times$

$e^{i\phi}: \mathbb{B} = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & 0 & \\ & 1 & 0 & & \\ & & 1 & 0 & \\ & 0 & & 1 & 0 \end{pmatrix} \qquad \mathbb{A}\mathbb{B} \overset{5\times5}{=} \begin{pmatrix} 0 & & & & \\ -i & 0 & & 0 & \\ & 0 & 0 & & \\ & & i & 0 & \\ & 0 & & 2i & 0 \end{pmatrix} \qquad \mathbb{B}\mathbb{A} \overset{5\times5}{=} \begin{pmatrix} 0 & & & & \\ -2i & 0 & & & \\ & -i & 0 & 0 & \\ & 0 & 0 & 0 & \\ & & & i & 0 \end{pmatrix}$

$\mathbb{A}\mathbb{B} - \mathbb{B}\mathbb{A} = \begin{pmatrix} 0 & & & & \\ i & 0 & & 0 & \\ & i & 0 & & \\ & 0 & i & 0 & \\ & & & i & 0 \end{pmatrix}$ OK

$\mathbb{A}\mathbb{S}_1, \quad \mathbb{A}^2\mathbb{S}_1 \qquad \mathbb{S}_1 \overset{5\times5}{=} \frac{1}{i\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad = -1\cdot\mathbb{S}_1$

$\mathbb{A}\mathbb{S}_1 = \begin{pmatrix} -2i & & & & \\ & -i & & 0 & \\ & & 0 & & \\ & 0 & & i & \\ & & & & 2i \end{pmatrix}\frac{1}{i\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{i\sqrt{2}}\begin{pmatrix} 0 \\ i \\ 0 \\ i \\ 0 \end{pmatrix} \qquad \mathbb{A}^2\mathbb{S}_1 = \begin{pmatrix} -2i & & & & \\ & -i & & 0 & \\ & & 0 & & \\ & 0 & & i & \\ & & & & 2i \end{pmatrix}\frac{1}{i\sqrt{2}}\begin{pmatrix} 0 \\ i \\ 0 \\ i \\ 0 \end{pmatrix} = \frac{1}{i\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$