21.3 Operátory

1) lineární $\hat{A}(u+v) = \hat{A}u + \hat{A}v$

$\hat{A}(cu) = c\hat{A}u \qquad c \in \mathbb{C}$

a) $\hat{A}u = du$ OK

$\hat{A}(u+v) = d(u+v) = du + dv = \hat{A}u + \hat{A}v$ ✓

$\hat{A}(cu) = dcu = cdu = c\hat{A}u$ ✓

b) $\hat{A}u = u^*$

$\hat{A}(cu) = c^*u^* = c^*\hat{A}(u) \neq c\hat{A}(u)$ ✗

c) $\hat{A}u = u^2$

$\hat{A}(u+v) = u^2 + 2uv + v^2 = A(u) + A(v) + 2uv \neq u^2 + v^2 = A(u) + A(v)$ ✗

$\hat{A}(cu) = c^2u^2 = c^2A^2(u) \neq cA(u)$ ✗

d) $\hat{A}(u) = \frac{du}{dx}$

$A(u+v) = \frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx} = A(u) + A(v)$ ✓

$A(cu) = \frac{d(cu)}{dx} = c\frac{du}{dx} = c\,A(u)$ ✓

e) $\hat{A}(u) = \frac{1}{u}$

$A(u+v) = \frac{1}{u+v} \neq \frac{1}{u} + \frac{1}{v} = A(u) + A(v)$ ✗

f) $\hat{A}\hat{B} = [\hat{H}, \hat{B}] = \hat{H}\hat{B} - \hat{B}\hat{H}$

$\hat{A}(\hat{B}+\hat{C}) = \hat{A}[\hat{H}, \hat{B}+\hat{C}] = [\hat{H}, \hat{B}] + [\hat{H}, \hat{C}] = \hat{A}\hat{B} + \hat{A}\hat{C}$ ✓

$\hat{A}c\hat{B} = [\hat{H}, c\hat{B}] = c[\hat{H}, \hat{B}] = c\hat{A}\hat{B}$ ✓

OKMT2-1b
cf. LHO
a^+, a

2) $(\hat{A}-\hat{B})(\hat{A}+\hat{B}) = \hat{A}^2 - \hat{B}^2 - \hat{B}\hat{A} + \hat{A}\hat{B} = \hat{A}^2 - \hat{B}^2 + [A,B]$

3) $[\sum_i \hat{A}_i, \sum_j \hat{A}_j] = \sum_i A_i \sum_j A_j - \sum_j A_j \sum_i A_i = \sum_{ij} (A_i A_j - A_j A_i) = \sum_{ij} [A_i, A_j]$

4) $[\hat{A}\hat{B}, \hat{C}]$ pomocí $[\hat{A}, \hat{C}]$ a $[\hat{B}, \hat{C}]$

$[\hat{A}\hat{B}, \hat{C}] = \underset{\pm ACB}{\hat{A}BC} - \underset{\mp ACB}{CAB} = ABC - CAB - ACB + ACB =$

$= A[B,C] + [A,C]B$

5) $A\psi = \dfrac{d^2\psi}{dx^2} + 3\psi^2$ lineární?

$A(\psi + \phi) = \dfrac{d^2}{dx^2}(\psi + \phi) + 3(\psi+\phi)^2 = \dfrac{d^2\psi}{dx^2} + \dfrac{d^2\phi}{dx^2} + 3\psi^2 + 3\phi^2 + \underbrace{6\psi\phi}_{\times}$

6) $\left(\dfrac{d}{dx} + x\right)^2$?

$\left(\dfrac{d}{dx} + x\right)^2 \psi = \left(\dfrac{d}{dx} + x\right)\left[\dfrac{d\psi}{dx} + x\psi\right] = \dfrac{d^2\psi}{dx^2} + x\dfrac{d\psi}{dx} + \dfrac{d}{dx}(x\psi) + x^2\psi$

$= \dfrac{d^2\psi}{dx^2} + x\dfrac{d\psi}{dx} + \psi + x\dfrac{d\psi}{dx} + x^2\psi = \dfrac{d^2\psi}{dx^2} + 2x\dfrac{d\psi}{dx} + (x^2+1)\psi$

$\left(\dfrac{d}{dx} + x\right)^2 = \dfrac{d^2}{dx^2} + 2x\dfrac{d}{dx} + x^2 + 1$

7) $\left(\dfrac{d}{dx} + \dfrac{1}{x}\right)^3 \psi = \left(\dfrac{d}{dx} + \dfrac{1}{x}\right)^2 \left(\dfrac{d\psi}{dx} + \dfrac{1}{x}\psi\right) = \left(\dfrac{d}{dx} + \dfrac{1}{x}\right)\left[\dfrac{d^2\psi}{dx^2} + \dfrac{d}{dx}\left(\dfrac{1}{x}\psi\right) + \dfrac{1}{x}\dfrac{d\psi}{dx} + \dfrac{1}{x^2}\psi\right]$

$= \left(\dfrac{d}{dx} + \dfrac{1}{x}\right)\left[\dfrac{d^2\psi}{dx^2} \cancel{-\dfrac{1}{x^2}\psi} + \dfrac{1}{x}\dfrac{d\psi}{dx} + \dfrac{1}{x}\dfrac{d\psi}{dx} \cancel{+\dfrac{1}{x^2}\psi}\right] = \left(\dfrac{d}{dx} + \dfrac{1}{x}\right)\left[\dfrac{d^2\psi}{dx^2} + \dfrac{2}{x}\dfrac{d\psi}{dx}\right]$

$= \dfrac{d^3\psi}{dx^3} + \dfrac{d}{dx}\left(\dfrac{2}{x}\dfrac{d\psi}{dx}\right) + \dfrac{1}{x}\dfrac{d^2\psi}{dx^2} + \dfrac{2}{x^2}\dfrac{d\psi}{dx} = \dfrac{d^3\psi}{dx^3} \cancel{-\dfrac{2}{x^2}\dfrac{d\psi}{dx}} + \dfrac{2}{x}\dfrac{d^2\psi}{dx^2} + \dfrac{1}{x}\dfrac{d^2\psi}{dx^2} \cancel{+\dfrac{2d}{x^2 dx}}$

$= \dfrac{d^3\psi}{dx^3} + \dfrac{3}{x}\dfrac{d^2\psi}{dx^2}$

$\left(\dfrac{d}{dx} + \dfrac{1}{x}\right)^3 = \dfrac{d^3}{dx^3} + \dfrac{3}{x}\dfrac{d^2}{dx^2}$

8) $\left(x\frac{d}{dx}\right)^2$ vs $\left(\frac{d}{dx}x\right)^2$

$$A\psi = \left(x\frac{d}{dx}\right)^2\psi = x\frac{d}{dx}\left(x\frac{d}{dx}\psi\right) = x\frac{d}{dx}\left(x\frac{d\psi}{dx}\right) = x\frac{d\psi}{dx} + x^2\frac{d^2\psi}{dx^2}$$

$$B\psi = \left(\frac{d}{dx}x\right)^2\psi = \frac{d}{dx}\left[x\frac{d}{dx}(x\psi)\right] = \frac{d}{dx}\left[x\psi + x^2\frac{d\psi}{dx}\right] = \psi + x\frac{d\psi}{dx} + 2x\frac{d\psi}{dx} + x^2\frac{d^2\psi}{dx^2} =$$

$$= \psi + 3x\frac{d\psi}{dx} + x^2\frac{d^2\psi}{dx^2}$$

$$A = \left(x\frac{d}{dx}\right)^2 = x\frac{d}{dx} + x^2\frac{d^2}{dx^2}$$

$$B = \left(\frac{d}{dx}x\right)^2 = 1 + 3x\frac{d}{dx} + x^2\frac{d^2}{dx^2}$$

? $\left(y\frac{d}{dx}\right)^2$ vs. $\left(\frac{d}{dx}y\right)^2$

$(yx)^2$ vs. $(xy)^2$

9) $(\vec{p} - q\vec{A})^2$ $\quad \vec{p} = -i\hbar\vec{\nabla}$ $\quad \vec{A} = \vec{A}(r,t)$

$$(p - qA)^2\psi = (-i\hbar\vec{\nabla} - q\vec{A})^2\psi = (-i\hbar\vec{\nabla} - q\vec{A})(-i\hbar\nabla\psi - q\vec{A}\psi)$$

$$= -\hbar^2\nabla^2\psi + i\hbar q\,\nabla\cdot(\vec{A}\psi) + i\hbar q\,\vec{A}\nabla\psi + q^2A^2\psi$$

$$\vec{\nabla} = \begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{pmatrix} \quad \vec{A} = \begin{pmatrix}A_x\\ A_y\\ A_z\end{pmatrix} \quad \psi = \begin{pmatrix}\psi\\ \psi\\ \psi\end{pmatrix} = \psi \quad (\vec{\nabla})(\vec{A}\psi) = \begin{pmatrix}\frac{\partial}{\partial x} & A_x\psi\\ \frac{\partial}{\partial y} & A_y\psi\\ \frac{\partial}{\partial z} & A_z\psi\end{pmatrix}^T = \sum_\alpha \frac{\partial}{\partial \alpha}A_\alpha\psi = \vec{\nabla}\cdot\vec{A}\psi$$

$$\frac{\partial}{\partial x}A_x\psi = \frac{\partial A_x}{\partial x}\psi + A_x\frac{\partial\psi}{\partial x}$$

$$\vec{\nabla}\cdot(\vec{A}\psi) = (\vec{\nabla}\cdot\vec{A})\psi + \vec{A}\nabla\psi$$

$$\rightarrow (\vec{p} - q\vec{A})^2 = -\hbar^2\nabla^2 + i\hbar q\,\nabla\cdot\vec{A} + 2i\hbar q\,\vec{A}\nabla + q^2A^2$$

⋮

10) $\left[\frac{d}{dx}, x\right]$

$$\left[\frac{d}{dx}, x\right]\psi = \left(\frac{d}{dx}x\psi - x\frac{d}{dx}\psi\right) = \psi + x\frac{d\psi}{dx} - \cancel{x\frac{d\psi}{dx}} = \psi$$

$$\left[\frac{d}{dx}, x\right] = 1$$

11) $[x, p] = \left[x, -i\hbar\frac{d}{dx}\right] = -i\hbar\left[x, \frac{d}{dx}\right]$

$$-i\hbar\left[x, \frac{d}{dx}\right]\psi = -i\hbar\left[x\frac{d\psi}{dx} - \frac{d}{dx}x\psi\right] = -i\hbar\left[x\frac{d\psi}{dx} - \cancel{x}\frac{d\psi}{dx} - \psi\right] = i\hbar\psi$$

$$[x, p] = i\hbar$$

13) $[x-p, x+p] = [x, \cancel{x}] + [x, p] - [p, x] - [p, \cancel{p}]$

$$= 2[x, p] = 2i\hbar$$

14) $[xp, x] = xpx - x^2p = x(px - xp) = x[p, x] = -i\hbar x$

15) $\left[\frac{\partial}{\partial x}, f(x,y,z)\right]$

$$\left[\frac{\partial}{\partial x}, f(x,y,z)\right]\psi = \frac{\partial}{\partial x}f\psi - f\frac{\partial}{\partial x}\psi = \frac{\partial f}{\partial x}\psi + \cancel{f\frac{\partial\psi}{\partial x}} - \cancel{f\frac{\partial\psi}{\partial x}} = \frac{\partial f}{\partial x}\psi$$

$$\left[\frac{\partial}{\partial x}, f(x,y,z)\right] = \frac{\partial f}{\partial x}$$

16) $[x, D^2]$

$$[x, D^2]\psi = xD^2\psi - D^2x\psi = xD^2\psi - \frac{\partial}{\partial x}\frac{\partial}{\partial x}x\psi = x\frac{\partial^2\psi}{\partial x^2} - \frac{\partial}{\partial x}\psi - \frac{\partial}{\partial x}x\frac{\partial\psi}{\partial x}$$

$$= \cancel{x\frac{\partial^2\psi}{\partial x^2}} - \frac{\partial\psi}{\partial x} - \frac{\partial\psi}{\partial x} - \cancel{x\frac{\partial^2\psi}{\partial x^2}} = -2\frac{\partial\psi}{\partial x}$$

$$[x, D^2] = -2\frac{\partial}{\partial x}$$

18) $\hat{T}_a \psi(x) = \psi(x+a)$

$$\psi(x+a) = \psi(x) + a\frac{d\psi(x)}{dx} + \frac{a^2}{2}\frac{d^2\psi(x)}{dx^2} + \dots = \sum_{n=0}^{\infty} \frac{a^n}{n!}\frac{d^n}{dx^n}\psi(x)$$

$$\hat{T}_a = e^{a\frac{d}{dx}}$$

21) Hermit conjugate: $\langle \psi | H\psi \rangle = \langle H^+\psi | \psi \rangle$

$$\int \psi^*(x)\, H\psi(x)\, dx \qquad \int [H^+\psi(x)]^*\, \psi(x)\, dx$$

$$H = \frac{\partial}{\partial x}$$

$$\int \psi^* \frac{\partial \phi}{\partial x} dx = \int_{-\infty}^{\infty} \frac{d}{dx}\left(\psi^* \phi\right) dx - \int \frac{d\psi^*}{dx}\phi\, dx = \left[\psi^*\phi\right]_{-\infty}^{\infty} - \int \frac{d\psi^*}{dx}\phi\, dx$$

$$\left[\psi^*\phi\right]_{-\infty}^{\infty} = 0$$

$$\int (uv')dx = uv - \int (u'v)dx$$

$$\langle \psi | A\phi \rangle = \int \psi^* \frac{\partial\phi}{\partial x} dx \qquad A = \frac{d}{dx}$$

$$\langle A^+\psi | \phi \rangle = -\int \frac{d\psi^*}{dx}\phi\, dx = -\int \left[\frac{d}{dx}\psi\right]^* \phi\, dx = \int \left[-\frac{d}{dx}\psi\right]^* \phi\, dx \qquad A^+ = -\frac{d}{dx}$$

22) $\frac{d^n}{dx^n}$

pro $n=2$
$$\int \psi^* \frac{d^2\phi}{dx^2} dx = \int \psi^* \frac{d}{dx}\left(\frac{d\phi}{dx}\right)dx = \int \frac{d}{dx}\left[\psi^* \frac{d\phi}{dx}\right]dx - \int \frac{d\psi^*}{dx}\frac{d\phi}{dx}dx$$

$\to$ v limitách $\to$ const $\stackrel{?}{=} 0$

$$= -\int \frac{d\psi^*}{dx}\frac{d\phi}{dx}dx = -\int \frac{d}{dx}\left[\frac{d\psi^*}{dx}\phi\right]dx + \int \frac{d^2\psi^*}{dx^2}\phi\, dx = \int \frac{d^2\psi^*}{dx^2}\phi\, dx$$

evidentně $\left(\frac{d^n}{dx^n}\right)^+ = (-)^n \left(\frac{d}{dx}\right)^n$

23) ~~$p^+ = i\hbar\nabla$~~ ~~p^+~~ $\quad p = -i\hbar\nabla$

$$\langle \psi | c\phi \rangle = \int \psi^* c\, \phi\, dx = \int c\, \psi^* \phi\, dx = \int (c^*\psi)^* \phi\, dx = \langle c^*\psi | \phi \rangle$$

$\uparrow$ complex constant

$$c^+ = c^*$$

$$p^+ = (-i\hbar)^*(-\nabla) = i\hbar(-\nabla) = -i\hbar\nabla \Rightarrow \text{Hermit.}$$

25) p v ~~kartézských~~ sférických sauradniciach   physics.stackexchange.com   T2-3b
How to construct the radial component.

? $\hat{p}\psi = -i\hbar \frac{1}{r}\frac{\partial}{\partial r}(r\psi)$ ok

? $\hat{D}\psi = -i\hbar \frac{\partial}{\partial r}\psi$ D^+ ?

$$\int_0^\infty r^2\psi^* (-i\hbar)\frac{1}{r}\frac{\partial}{\partial r}(r\varphi)\,dr = -i\hbar\int_0^\infty \psi^* r \frac{\partial}{\partial r}(r\varphi)\,dr =$$

$$= -i\hbar \int_0^\infty \frac{d}{dr}[\psi^* r\, r\varphi]\,dr + i\hbar\int_0^\infty \frac{\partial \psi^* r}{dr}\, r\varphi\, dr$$

$$= -i\hbar\,[\psi^* r^2 \varphi]_0^\infty + i\hbar\int_0^\infty \frac{d(\psi^* r)}{dr}\, r\varphi\, dr =$$

$\infty : \psi^* \,\&\, \varphi = 0$

$0 : r = 0$

$$= \int_0^\infty \left[\frac{i\hbar\, d(\psi^* r)}{dr}\right] r\varphi\, dr = \int_0^\infty \left[\frac{-i\hbar\, d(\psi r)}{r\, dr}\right]^* r^2\varphi\, dr$$

$$p\psi = -\frac{i\hbar\, d(\psi r)}{r\,dr} = p\psi \quad \text{OK, Hermit}$$

$D = -i\hbar\frac{d}{dr}$ $D^+ = ?$

$$\int_0^\infty r^2\psi^*(-i\hbar)\frac{\partial}{\partial r}\varphi\, dr = (-i\hbar)\int_0^\infty r^2\psi^*\frac{\partial}{\partial r}\varphi\, dr = (-i\hbar)\int_0^\infty \frac{d}{dr}[r^2\psi^*\varphi]\,dr + i\hbar\left[\int_0^\infty \frac{d(r^2\psi^*)}{dr}\varphi\, dr\right] =$$

$$= i\hbar\int_0^\infty \left[2r\psi^* + r^2\frac{d\psi^*}{dr}\right]\varphi\, dr = \int_0^\infty \left[-i\hbar 2r\psi - i\hbar r^2\frac{d\psi}{dr}\right]^*\varphi\, dr =$$

$$\cancel{\left[-i\hbar\frac{d}{dr}\right]^+ = \left[-i\hbar 2r - i\hbar r^2\frac{d}{dr}\right]}$$

$$= \int_0^\infty r^2\left[-\frac{i\hbar 2}{r}\psi - i\hbar\frac{d\psi}{dr}\right]^*\varphi\, dr$$

$$D^+ = \left[-i\hbar\frac{d}{dr}\right]^+ = \left[-i\hbar\left(\frac{2}{r}+\frac{d}{dr}\right)\right] = D' \quad \to \text{nehermitovský}$$

Hermitise: $\frac{D + D'}{2} = \frac{-i\hbar\left[\frac{d}{dr} + \frac{2}{r} + \frac{d}{dr}\right]}{2} = -i\hbar\left[\frac{d}{dr} + \frac{1}{r}\right]$

$$-i\hbar\left[\frac{d}{dr}+\frac{1}{r}\right]\psi = -\frac{i\hbar}{r}\frac{d}{dr}(r\psi) = \hat{p}_r$$

28) $\left[e^{i\alpha\frac{\partial}{\partial\varphi}}\right]^+$? $\alpha\in R$

$$e^{i\alpha\frac{\partial}{\partial\varphi}} = \sum_{n=0}^{\infty} \frac{\left(i\alpha\frac{\partial}{\partial\varphi}\right)^n}{n!}$$

$$(i\alpha)^+ = (i\alpha)^* = -i\alpha$$

$$\left(\frac{\partial}{\partial\varphi}\right)^+ = -\frac{\partial}{\partial\varphi}$$

$$\left[e^{i\alpha\frac{\partial}{\partial\varphi}}\right]^+ = \left[\sum_{n=0}^{\infty} \frac{\left(i\alpha\frac{\partial}{\partial\varphi}\right)^n}{n!}\right]^+ = \sum_{n=0}^{\infty} \frac{(-i\alpha)^n\left(-\frac{\partial}{\partial\varphi}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{\left(i\alpha\frac{\partial}{\partial\varphi}\right)^n}{n!} = e^{i\alpha\frac{\partial}{\partial\varphi}}$$

30) $[\hat{A}\hat{B}]^+ = ?$

$$\int \varphi^* (\hat{A}\hat{B}\varphi)\,d\tau = \int (A^+\varphi)^* B\varphi\,d\tau = \int (B^+A^+\varphi)^*\varphi\,d\tau \quad \text{OK}$$

$$[\hat{A}\hat{B}]^+ = B^+A^+$$

31) $\langle A^2\rangle \geq 0$ A - op. veličiny - Hermitovský / sym.

$$\int \psi^* A^2 \psi\,d\tau = \int (A\psi)^* A\psi\,d\tau = \int |A\psi|^2 d\tau \geq 0 \quad \text{OK}$$

32) $A, B: [\hat{A},\hat{B}] = 1$

$$[A, B^2] = AB^2 - B^2A = AB\underbrace{B - BAB}\ \underbrace{+BAB - B}BA = [A,B]B + B[A,B]$$

$ABB \quad - BBA$

$ABB - BAB \quad - BBA + BAB$

$$= 2B$$

33) $[A, B^n] = AB^n - B^nA = ABB^{n-1} - BAB^{n-1} + BAB^{n-1} - BB^{n-1}A$

$[A, B^n] = n B^{n-1}$ $ABB^{n-1} - BAB^{n-1} - B^{n-1}BA + BAB^{n-1}$

$$= [A,B]B^{n-1} + B[A,B^{n-1}] = B^{n-1} + (n-1)B^{n-1} = nB^{n-1}$$

důk. indukcí $= (n-1)B^{n-2}$

$[A, f(B)]$ $f(B) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\hat{B}^n$ $f^{(n)} = \frac{d^n f}{dx^n}$

$$[A, f(B)] = \left[A, \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}B^n\right] = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}[A, B^n] =$$

$$= \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\, nB^{n-1} = \sum_{n=1}^{\infty}\frac{f^{(n)}(0)}{(n-1)!}B^{n-1} = \sum_{m=0}^{\infty}\frac{f^{(m+1)}(0)}{m!}B^m$$

$m = n-1$
$n = m+1$

$$= \sum_{m=0}^{\infty}\frac{\frac{d^m}{dx^m}\frac{d f}{dx}}{m!}B^m = \sum_{m=0}^{\infty}\frac{\frac{d^m}{dx^m}f'}{m!}B^m = f'(B)$$

21.5 Vlnové fce

2) $\psi(x)$ & $\exp[i f(x)]\,\psi(x) = \varphi$

$$j(\psi) = \frac{\hbar}{2mi}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right)$$

$$j(\varphi) = \frac{\hbar}{2mi}\left(\exp[-if(x)]\,\psi^*(x)\frac{\partial \exp[if(x)]\,\psi(x)}{\partial x} - \exp[if(x)]\,\psi(x)\frac{\partial \exp[-if(x)]\,\psi^*(x)}{\partial x}\right]$$

$$= \frac{\hbar}{2mi}\left[\exp[-if(x)]\,\psi^*(x)\left[\psi(x)\frac{\partial e^{if(x)}}{\partial x} + e^{if(x)}\frac{\partial \psi(x)}{\partial x}\right] - e^{if(x)}\psi(x)\left[\frac{\partial e^{-if(x)}}{\partial x}\psi^* + e^{-if(x)}\frac{\partial \psi^*}{\partial x}\right]\right]$$

$$= \frac{\hbar}{2mi}\left[\psi^*(x)\frac{\partial \psi(x)}{\partial x} - \psi(x)\frac{\partial \psi^*}{\partial x}\right]$$

$$+ \frac{\hbar}{2mi}\left[i\psi^*(x)\,\psi(x)\frac{\partial f(x)}{\partial x} + i\,\psi(x)\,\psi^*(x)\frac{\partial f(x)}{\partial x}\right]$$

$$= j(\psi) + \frac{\hbar}{m}\,\psi^*(x)\,\psi(x)\frac{\partial f(x)}{\partial x}$$

4) Fourier $\psi(x) = h\exp(ik_0 x)$ pro $-a/2 \le x \le a/2$; $\psi(x) = 0$ jinak

$$\varphi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \psi(x)\, e^{-ikx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a/2}^{a/2} h e^{ik_0 x} e^{-ikx}\,dx = \frac{h}{\sqrt{2\pi}}\int_{-a/2}^{a/2} e^{i(k_0-k)x}\,dx$$

$$= \frac{h}{\sqrt{2\pi}}\,\frac{1}{i(k_0-k)}\left[e^{i(k_0-k)x}\right]_{-a/2}^{a/2} = \frac{h}{\sqrt{2\pi}\,i(k_0-k)}\left[e^{i(k_0-k)a/2} - e^{-i(k_0-k)a/2}\right] =$$

$\exp(ix) = \cos(x) + i\sin(x)$ $\rightarrow$ $\sin(x) = \frac{1}{2i}[\exp(ix) - \exp(-ix)]$

$\exp(-ix) = \cos(x) - i\sin(x)$

$$= \frac{2h}{\sqrt{2\pi}}\,\frac{\sin[(k_0-k)\,a/2]}{k_0-k}$$

$\rightsquigarrow \operatorname{sinc}[(k_0-k)a/2]$, Bessel J_0

21.6 Vl. fce

UKMTZ-6

2) $H\psi = E\psi$ $\quad H$ - reálný

c.c.: $H\psi^* = E\psi^*$

$Re(\psi) = \frac{\psi + \psi^*}{2}$ - reálná část ψ

$Im(\psi) = \frac{\psi - \psi^*}{2i}$ - imaginární část ψ

$H\psi + H\psi^* = E\psi + E\psi^*$

$H\left(\frac{\psi+\psi^*}{2}\right) = E\left(\frac{\psi+\psi^*}{2}\right)$

3) $\psi(x) = A\cos(kx)$; $k \in \mathbb{R}$, vl. fce $\hat{p}$?

$\cancel{p(x)}\ p\psi = -i\hbar \frac{d}{dx} A\cos(kx) = +i\hbar Ak\sin(kx) \neq C\cos(kx)$

p^2

$p^2\psi = -\hbar^2 \frac{d^2}{dx^2} A\cos(kx) = +\hbar^2 k^2 A\cos(kx) = +\hbar^2 k^2 \psi$ OK

$\rightarrow p = \hbar k$ vl. číslo

$\psi(x) \cancel{A} = A\cos(kx) + B\sin(kx)$

$p\psi = -i\hbar \frac{d}{dx}\left[A\cos(kx) + B\sin(kx)\right] = i\hbar Ak\sin(kx) - i\hbar Bk\cos(kx)$

$= i\hbar k\left[A\sin(kx) - B\cos(kx)\right]$

$i A = B$ $\quad$ $i \; 1$

$-i B = A$ $\quad$ $1 \; i$

say $A = 1$, $B = i$

$\rightarrow i\hbar k\sin(kx) + \hbar k\cos(kx)$

$\hbar k\left[\cos(kx) + i\sin(kx)\right] \rightarrow \hbar k\, e^{ikx}$

8) vl. čísla a fce op. $-i\frac{d}{d\varphi} = \hat{A}$ ~~φ~~ φ ve sférických souřadnicích → atom vodíku, ..., rotace, ...

$$\hat{A}\psi = \lambda\psi$$

$$-i\frac{d\psi}{d\varphi} = \lambda\psi$$

$$\frac{d\psi}{d\varphi} = i\lambda\psi \quad \Rightarrow \text{uhodneme } \psi = e^{i\lambda\varphi}$$

! spojitost : $\psi(\varphi) = \psi(\varphi + 2\pi)$

$$e^{i\lambda\varphi} = e^{i\lambda(\varphi+2\pi)}$$

$$1 = e^{i2\pi\lambda} \Rightarrow \lambda = 0, \pm 1, \pm 2 \dots$$

normování $\psi = N e^{i\lambda\varphi}$ normování

$$\int_0^{2\pi} \psi^* \psi \, d\varphi = \int_0^{2\pi} N^2 e^{-i\lambda\varphi} e^{i\lambda\varphi} d\varphi = N^2 \int_0^{2\pi} d\varphi = 2\pi N^2 \overset{\Downarrow}{=} 1$$

$$N = \frac{1}{\sqrt{2\pi}}$$

$$\rightarrow \cancel{\psi = \frac{1}{\sqrt{2\pi}} e^{i\lambda\varphi}} \qquad \psi = \frac{1}{\sqrt{2\pi}} e^{i\lambda\varphi}$$

9) $\hat{A} = \sin\left(\frac{d}{d\varphi}\right)$ vl. čísla a fce, φ - úhel rotace ve sfér. souř.

$$A\psi = \lambda\psi$$

$$\sin\left(\frac{d}{d\varphi}\right)\psi = \lambda\psi \qquad \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{d^{2n+1}\psi}{d\varphi^{2n+1}} = \lambda\psi$$

tip $\psi = N e^{\alpha\varphi}$ $\quad \frac{d^{2n+1}}{d\varphi^{2n+1}} N e^{\alpha\varphi} = N e^{\alpha\varphi} \alpha^{2n+1}$

$$\psi \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \alpha^{2n+1}}_{\sin(\alpha)} = \lambda\psi$$

$$\lambda = \sin(\alpha)$$

$$N e^{\alpha(\varphi+2\pi)} = \psi(\varphi) = \psi(\varphi + 2\pi) = N e^{\alpha(\varphi+2\pi)}$$

$$\rightarrow \alpha = im \qquad m = 0, \pm 1, \dots$$

$$\lambda = \sin(im) = i\sinh(m)$$