mom. hybnosti contd. x rotátor T8.1

Rotátor / 1

- tuhé těleso s mom. setrvačnosti J rotuje okolo osy z ve vzdálenosti r

$$H = -\frac{\hbar^2}{2m}\Delta = \left[\cancel{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)} + \underbrace{\frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}} + \cancel{\frac{\partial^2}{\partial z^2}}\right]\cdot\frac{\hbar^2}{2m}$$

válcové souřadnice

$$= -\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} = -\frac{\hbar^2}{2J}\frac{\partial^2}{\partial\varphi^2}$$

$\varphi \in [0, 2\pi]$

$$\rightarrow H\psi = E\psi$$

$$-\frac{\hbar^2}{2J}\frac{\partial^2\psi}{\partial\varphi^2} = E\psi$$

$$\psi = A e^{ic\varphi}$$

$$\frac{\hbar^2}{2J}c^2\psi = E\psi \qquad c = \pm\sqrt{\frac{2EJ}{\hbar^2}}$$

$\psi(\varphi) = \psi(\varphi + 2\pi)$ ← spojitost

$$e^{ic\varphi} = e^{ic(\varphi + 2\pi)}$$

$$\cancel{e^{ic\varphi} = e^{ic\varphi}e^{2\pi i c}} \qquad e^{ic\varphi} = e^{ic\varphi}e^{ic2\pi}$$

$$\cancel{2\pi i c = n}$$

$$\rightarrow 2\pi c = 2\pi n$$

$(c = n \leftarrow \in \mathbb{Z}$

pokud se posunu o 2π, musí být fáze násobkem $2\pi n$, aby $e^{i2\pi n}$ bylo $= 1$

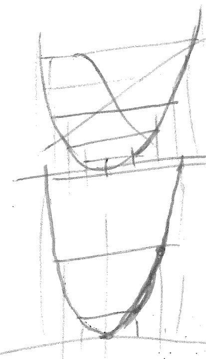
$$c = \pm\sqrt{\frac{2EJ}{\hbar^2}} = n$$

$$\frac{2EJ}{\hbar^2} = n^2 \quad \Rightarrow E = \frac{\hbar^2 n^2}{2J}$$

$$\rightarrow \psi = A e^{in\varphi} \qquad E_n = \frac{\hbar^2 n^2}{2J}$$

$$A^2\int_0^{2\pi} e^{-in\varphi}e^{in\varphi}\,d\varphi = A^2\int_0^{2\pi} d\varphi = A^2[\varphi]_0^{2\pi} = 2\pi A^2$$

$$1 = A^2 2\pi \rightarrow A = \frac{1}{\sqrt{2\pi}}$$



← energie jako volné částice

2x degenerované

nebo jako jáma, ale jáma nedegenerované (proč?)

12 $\quad\times\quad$ rotátor $\quad$ Tf.15

$$\psi = \frac{1}{\sqrt{2\pi}} e^{in\varphi}$$

určete $\langle L_z \rangle$ a $\langle L_z^2 \rangle$, $L_z = -i\hbar \frac{\partial}{\partial\varphi}$, $L_z^2 = -\hbar^2 \frac{\partial^2}{\partial\varphi^2}$

$$\langle L_z \rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\varphi}\left(-i\hbar\frac{\partial}{\partial\varphi}\right) e^{in\varphi} d\varphi = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\varphi}(-i\hbar\, in) e^{in\varphi} d\varphi$$

$$= \frac{\hbar n}{2\pi}\int_0^{2\pi} d\varphi = \hbar n$$

$$\langle L_z^2 \rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\varphi}\left(-\hbar^2\frac{\partial^2}{\partial\varphi^2}\right) e^{in\varphi} d\varphi = \hbar^2 n^2 \qquad \text{OK}$$

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čas. vývoj $\quad \psi(t=0) = A\sin^2\varphi$

$$\varphi_n = \frac{1}{\sqrt{2\pi}} e^{i\varphi n}$$

$$\langle \psi_n | \psi \rangle = \frac{A}{\sqrt{2\pi}}\int_0^{2\pi} e^{-i\varphi n} \sin^2\varphi \, d\varphi$$

$$\sin\varphi = \frac{1}{2i}\left[e^{i\varphi} - e^{-i\varphi}\right]$$

$$\sin^2\varphi = -\frac{1}{4}\left[e^{i\varphi} - e^{-i\varphi}\right]^2 = -\frac{1}{4}\left[e^{2i\varphi} + e^{-2i\varphi} - 2\right] =$$

$$= \frac{1}{2}\left[1 - \cos(2\varphi)\right]$$

$$\langle \psi_n | \psi \rangle = \frac{A}{\sqrt{2\pi}}\int_0^{2\pi} e^{-i\varphi n}\left[-\frac{1}{4}\left(e^{2i\varphi} + e^{-2i\varphi} - 2\right)\right] d\varphi$$

$$\cancel{A}\int_0^{2\pi} e^{-i\varphi n} e^{2i\varphi} d\varphi \overset{0 \text{ kro } n\neq 2}{=} \int_0^{2\pi} e^{-2i\varphi} e^{2i\varphi} d\varphi = 2\pi$$

$$\langle \psi_2 | \psi \rangle = \frac{A}{\sqrt{2\pi}}\left(-\frac{1}{4}\right) 2\pi = -\frac{\pi A}{2\sqrt{2\pi}} = -\frac{A}{2}\sqrt{\frac{\pi}{2}}$$

$$E_n = \frac{\hbar^2 n^2}{J2}$$

$$e^{\frac{E_n t}{i\hbar}} = e^{-\frac{i\hbar n^2}{2J}}$$

$$\psi = -\frac{A}{2}\sqrt{\frac{\pi}{2}}|2\rangle - \frac{A}{2}\sqrt{\frac{\pi}{2}}|-2\rangle + A\sqrt{\frac{\pi}{2}}|0\rangle$$

$$\psi(t) = -\frac{A}{2}\sqrt{\frac{\pi}{2}}|2\rangle e^{-\frac{i\hbar 4}{2J}} - \frac{A}{2}\sqrt{\frac{\pi}{2}}|-2\rangle e^{-\frac{i\hbar 4}{2J}} + A\sqrt{\frac{\pi}{2}}|0\rangle e^{0}$$

$$= -\frac{A}{2}\sqrt{\frac{\pi}{2}}|2\rangle e^{-\frac{2i\hbar}{J}} - \frac{A}{2}\sqrt{\frac{\pi}{2}}|-2\rangle e^{-\frac{2i\hbar}{J}} + A\sqrt{\frac{\pi}{2}}|0\rangle$$

$$= -\frac{A}{2}\sqrt{\frac{\pi}{2}}\frac{1}{\sqrt{2\pi}} e^{2i\varphi} e^{-\frac{2i\hbar}{J}} - \frac{A}{2}\sqrt{\frac{\pi}{2}}\frac{1}{\sqrt{2\pi}} e^{-2i\varphi} e^{-\frac{2i\hbar}{J}} + A\sqrt{\frac{\pi}{2}}\frac{1}{\sqrt{2\pi}}$$

$$= \frac{A}{2} - \frac{A}{4}\left[e^{2i\varphi} + e^{-2i\varphi}\right] e^{-\frac{2i\hbar}{J}}$$

$$= \frac{A}{2}\left[1 - \cos(2\varphi) e^{-\frac{2i\hbar}{J}}\right] \qquad \text{OK}$$

$\langle 0|L_x|-1\rangle$ A $L_+ = L_x + iL_y$ $L_- = L_x - iL_y$ Tf. 2

$\langle 0|\frac{1}{2}(L_+ + L_-)|-1\rangle$ $L_x = \frac{1}{2}(L_+ + L_-)$ $L_\pm|l,m\rangle = \hbar\sqrt{l(l+1) - m(m\pm1)}\,|l,m\pm1\rangle$

$L_y = \frac{1}{2i}(L_+ - L_-)$

$$= \frac{1}{2}\langle 0|L_+|-1\rangle + \frac{1}{2}\langle 0|L_-|-1\rangle =$$

$$= \frac{1}{2}\langle 0|\hbar\sqrt{2+0}|0\rangle + \frac{1}{2}\langle 0|\hbar\cancel{\sqrt{2+1(-2)}}|\overset{\text{"}-2\text{"}}{\cancel{-1}}\rangle = \frac{1}{2}\hbar\sqrt{2} = \frac{\hbar}{\sqrt{2}}$$

$$\frac{1}{2}(L_+ + L_-)|0\rangle = \frac{1}{2}L_+|0\rangle + \frac{1}{2}L_-|0\rangle =$$

$$= \frac{1}{2}\hbar\sqrt{2}|1\rangle + \frac{1}{2}\hbar\sqrt{2}|-1\rangle$$

$$\langle 1|\frac{1}{2}(L_+ + L_-)|0\rangle = \frac{\hbar}{\sqrt{2}}$$

$$\langle -1|\frac{1}{2}(L_+ + L_-)|0\rangle = \frac{\hbar}{\sqrt{2}}$$

$$\rightarrow (L_x)_{mn} = \frac{\hbar}{\sqrt{2}}\begin{array}{c} \begin{array}{ccc} -1 & 0 & 1 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array}\begin{array}{c} \\ -1 \\ 0 \\ 1 \end{array}$$

$$L_y|-1\rangle = \frac{1}{2i}(L_+ - L_-)|-1\rangle = \frac{1}{2i}\hbar\sqrt{2}|0\rangle - \frac{1}{2i}0$$ +B

$$L_y|0\rangle = \frac{1}{2i}(L_+ - L_-)|0\rangle = \frac{1}{2i}\hbar\sqrt{2}|1\rangle - \frac{1}{2i}\hbar\sqrt{2}|-1\rangle$$

$$L_y|1\rangle = \frac{1}{2i}(L_+ - L_-)|1\rangle = \frac{1}{2i}0 - \frac{1}{2i}\hbar\sqrt{2}|0\rangle$$

$$\langle m|L_y|n\rangle \rightarrow \frac{\hbar}{2\sqrt{2}}\begin{array}{c} \begin{array}{ccc} -1 & 0\ (|n\rangle) & 1 \end{array} \\ \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & +1 & 0 \end{pmatrix} \end{array}\begin{array}{c} \\ -1 \\ 0 \\ 1 \end{array} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & i & 0 \\ -i & 0 & i \\ 0 & -i & 0 \end{pmatrix}$$

$\langle m|$

$$L_xL_y - L_yL_x = \frac{\hbar^2}{2}\left[\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & i & 0 \\ -i & 0 & i \\ 0 & -i & 0 \end{pmatrix} - \begin{pmatrix} 0 & i & 0 \\ -i & 0 & i \\ 0 & -i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\right] =$$

$$= \frac{\hbar^2}{2}\left[\begin{pmatrix} -i & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & i \end{pmatrix} - \begin{pmatrix} i & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & -i \end{pmatrix}\right] = \frac{\hbar^2}{2}\begin{bmatrix} -2i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2i \end{bmatrix} =$$

$$= i\hbar^2\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = i\hbar\,\hbar\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$L_z|l,m\rangle = \hbar m|l,m\rangle$$

$$\langle l,m'|L_z|l,m\rangle = \hbar m\,\delta_{mm'} \qquad L_z = \hbar\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$ $\quad$ X $e^{i\theta} = \cos\theta + i\sin\theta$

TR. 3

$e^{-i\theta} = \cos\theta - i\sin\theta$

$\left[\frac{1}{2i}(e^{i\theta} - e^{-i\theta})\right]^3$ $\quad$ $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$ ok

$$= \frac{1}{(2i)^3}(e^{i\theta} - e^{-i\theta})\left[e^{2i\theta} + e^{-2i\theta} - 2\right] =$$

$$= -\frac{1}{8i}\left\{\left[(e^{i\theta} - e^{-i\theta})(-2) + (e^{3i\theta} + e^{-i\theta} - e^{i\theta} - e^{-3i\theta})\right]\right.$$

$$= \frac{1}{4i}(e^{i\theta} - e^{-i\theta}) - \frac{1}{8i}(e^{3i\theta} - e^{-3i\theta}) + \frac{1}{8i}(e^{i\theta} - e^{-i\theta})$$

$$(A-B)^3 = (A-B)(A^2 - 2AB + B^2) = A^3 - 2A^2B + AB^2 - A^2B + 2AB^2 - B^3 =$$
$$= A^3 - 3A^2B + 3AB^2 - B^3$$

$$= -\frac{1}{4(2i)}(e^{3i\theta} - e^{-3i\theta}) + \frac{3}{4(2i)}(e^{i\theta} - e^{-i\theta})$$
$$= -\frac{1}{4}\sin(3\theta) + \frac{3}{4}\sin(\theta)$$

X D

§ stavy $l_z = -1,\ 0,\ 1$ $\quad$ $Y_{l(-m)} = (-1)^m Y_{lm}^*$

$|11\rangle = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$ $\quad$ $|10\rangle = \sqrt{\frac{3}{4\pi}}\cos\theta$ $\quad$ $|1\,{-1}\rangle = +\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\phi}$

§ ~~Q11~~
- s pomocí L_+, L_-
- střední hodnoty, maticová reprez.

$$L_x = i\hbar\left(\sin\varphi\frac{\partial}{\partial\theta} + \cos\varphi\,\mathrm{cotg}\,\theta\frac{\partial}{\partial\varphi}\right)$$
$$L_y = i\hbar\left(-\cos\varphi\frac{\partial}{\partial\theta} + \sin\varphi\,\mathrm{cotg}\,\theta\frac{\partial}{\partial\varphi}\right)$$

$$L_+ = L_x + iL_y = i\hbar\left(i\sin\varphi\frac{\partial}{\partial\theta} + \cos\varphi\frac{\partial}{\partial\theta} + i\cos\varphi\,\mathrm{cotg}\,\theta\frac{\partial}{\partial\varphi} + i\,i\sin\varphi\,\mathrm{cotg}\,\theta\frac{\partial}{\partial\varphi}\right)$$
$$= \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\,\mathrm{cotg}\,\theta\frac{\partial}{\partial\varphi}\right)$$

$$L_- = L_x - iL_y = \hbar\left(i\sin\varphi\frac{\partial}{\partial\theta} - \cos\varphi\frac{\partial}{\partial\theta} + i\cos\varphi\,\mathrm{cotg}\frac{\partial}{\partial\varphi} - i\sin\varphi\,\mathrm{cotg}\,\theta\frac{\partial}{\partial\varphi}\right)$$
$$= \hbar e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + i\,\mathrm{cotg}\,\theta\frac{\partial}{\partial\varphi}\right)$$

TJ. 35

$$L_- |11\rangle = \hbar e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + i \cot\theta \frac{\partial}{\partial\varphi}\right)\left(-\sqrt{\frac{3}{8\pi}}\right)\sin\theta\, e^{i\phi}$$

D

$$\left(-\sqrt{\frac{3}{8\pi}}\right)\hbar\, \cancel{e^{-i\phi}}\left[(-\cos\theta)\cancel{e^{i\phi}} + i\cot\theta \sin\theta\,(i)\, \cancel{e^{i\phi}}\right]$$

opačné znaménko pro L_+ $\frac{3}{8} \Rightarrow 0$

$$= \left(+\sqrt{\frac{3}{8\pi}}\hbar\right)\left[+\cos\theta + \cos\theta\right] = \sqrt{\frac{3}{2\pi}}\hbar\cos\theta$$

$$L_- |11\rangle = \sqrt{2}\hbar\, |10\rangle$$

$$\sqrt{2}\hbar|10\rangle = \sqrt{\frac{3}{2\pi}}\hbar\cos\theta$$

$$|10\rangle = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$\langle 0 | L_x | -1\rangle = \int_0^{2\pi}\int_0^{\pi} d\phi\, d\theta \sqrt{\frac{3}{4\pi}}\cos\theta\left[i\hbar\left(\sin\varphi\frac{\partial}{\partial\theta} + \cos\varphi\cot\theta\frac{\partial}{\partial\varphi}\right)\right]\left(-\sqrt{\frac{3}{8\pi}}\right)\sin\theta\, e^{i\phi}$$

$\sin\theta$ Jacobi

$$= -\frac{3i\hbar}{4\sqrt{2}\pi}\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta\cos\theta\left[\sin\varphi\cos\theta\, e^{i\phi} + \cos\varphi\cot\theta\sin\theta\,(i)\, e^{i\phi}\right]$$

$$= -\frac{3i\hbar}{4\sqrt{2}\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \sin\theta\cos^2\theta\sin\varphi\, e^{i\phi} + i\cos^2\theta\sin\theta\cos\varphi\, e^{i\phi}$$

$$= -\frac{3\hbar}{4\sqrt{2}\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \cos^2\theta\sin\theta\; e^{i\phi}\left[i\sin\varphi - \cos\varphi\right]$$

$$= -\frac{3\hbar}{4\sqrt{2}\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \cos^2\theta\sin\theta\; e^{i\phi}\left(-e^{-i\phi}\right)$$

$$= \frac{3\hbar}{4\sqrt{2}\pi}\, 2\pi\left(-\frac{\cos^3\theta}{3}\Big|_0^{\pi}\right)$$

$$= \frac{3\hbar\, 2}{4\sqrt{2}\pi}\,\frac{2}{3} = \frac{\hbar}{\sqrt{2}}$$

OK

$\langle lm | L_x^2 + L_y^2 + L_z^2 | lm \rangle$ × C $\langle lm | L^2 | lm \rangle$ Tel. 4

$L_x = \frac{1}{2}(L_+ + L_-)$ $L_x^2 = \frac{1}{4}(L_+ + L_-)(L_+ + L_-) =$

$L_y = \frac{1}{2i}(L_+ - L_-)$ $= \frac{1}{4}(L_+^2 + L_-^2 + L_+L_- + L_-L_+)$

$L_y^2 = \frac{1}{4(i)^2}(L_+^2 + L_-^2 - L_+L_- - L_-L_+)$

$\cancel{L^2 = L_z^2 + \frac{1}{2}(L_+^2 + L_-^2)}$ $= -\frac{1}{4}(L_+^2 + L_-^2 - L_+L_- - L_-L_+)$

$L^2 = L_z^2 + \frac{1}{2}(L_+L_- + L_-L_+)$

$\langle lm | L^2 | lm \rangle = \langle lm | L_z^2 + \frac{1}{2}(L_+L_- + L_-L_+) | lm \rangle$

$= \hbar^2 \langle lm | \Big[m^2 | lm \rangle + \frac{1}{2}\Big(L_+ \big(\sqrt{l(l+1) - m(m-1)}\, | l\, m-1 \rangle\big)\Big)$

$+ \frac{1}{2}\Big(L_- \sqrt{l(l+1) - m(m+1)}\, | l\, m+1 \rangle\Big)\Big)$

$= \hbar^2 m^2 + \hbar^2 \frac{1}{2} \sqrt{l(l+1) - (m-1)m}\, \sqrt{l(l+1) - m(m-1)}\, | lm \rangle$

$+ \hbar^2 \frac{1}{2} \Big(\sqrt{l(l+1) - (m+1)m}\Big)^2 | lm \rangle$

$= \hbar^2 \Big[m^2 + \frac{1}{2}\big[l(l+1) - m(m-1) + l(l+1) - m(m+1) \big]\Big]$

$= \hbar^2 \Big[m^2 + \frac{1}{2}\big[2l(l+1) - 2m^2 \big]\Big] = \hbar^2 l(l+1)$ OK

$l^2 + \frac{1}{2}l^2 + \frac{1}{2}l - \frac{1}{2}l^2 + \frac{1}{2}l = l^2 + l = l(l+1)$ OK